Inhibition of chaotic escape from a potential well by incommensurate escape-suppressing excitations

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Theoretical results are presented concerning the reduction of chaotic escape from a potential well by means of a harmonic parametric excitation that satisfies an ultrasubharmonic resonance condition with the escape-inducing excitation. The possibility of incommensurate escape-suppressing excitations is demonstrated by studying rational approximations to the irrational escape-suppressing frequency. The analytical predictions for the suitable amplitudes and initial phases of the escape-suppressing excitation are tested against numerical simulations based on a high-resolution grid of initial conditions. These numerical results indicate that the reduction of escape is reliably achieved for small amplitudes and at, and only at, the predicted initial phases. For the case of irrational escape-suppressing frequencies, the effective escape-reducing initial phases are found to lie close to the accumulation points of the set of suitable initial phases that are associated with the complete series of convergents up to the convergent giving the chosen rational approximation.

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I. INTRODUCTION

Incidental escape from a potential well is a ubiquitous phenomenon in the physical sciences and engineering. However, the performance of a specific nonlinear system with a potential well subjected to a periodic excitation is often considered optimal if it operates in a periodic mode (i.e., inside the well) [1]. Recently, the application of weak parametric excitations (PEs) has been shown to be an effective technique for suppressing chaotic escape [2–4]. That theoretical work focused on the case of subharmonic resonance between the two driving frequencies involved, $\Omega = p\omega$, where Ω and ω are the escape-suppressing and escape-inducing frequencies, respectively. However, a number of experimental [5], theoretical [6–8], and numerical [9] studies of diverse dynamical systems show that chaos can be reliably eliminated by other nonsubharmonic resonances. The purpose of this present work is to discuss the inhibition of chaotic escape for nonsubharmonic resonances, and thence to approach the case of incommensurate escape-suppressing excitations by means of a series of ever better rational approximations, which are the successive convergents of the infinite continued fraction associated with the irrational ratio Ω/ω.

Since the coexistence of infinitely many periodic unstable solutions is today considered synonymous with chaos, we may test such a possibility by using the model of an unstable limit cycle affected by two weak harmonic perturbations which satisfy an ultrasubharmonic resonance condition:

$$x_{n+1} = [\mu + \varepsilon(f_n + \eta g_n)]x_n, \tag{1}$$

with $\mu > 1$, $\eta < 1$, $f_n = \sqrt{2}\cos n$, and $g_n = \sqrt{2}\cos(pn/q)$, $q > 1$ $(p \neq q)$. A similar recursion relation with $\eta = 0$ is considered in Ref. [9]. Note that $\langle f_n \rangle = \langle g_n \rangle = \langle f_n g_n \rangle = 0$ and $\langle f_n^2 \rangle = \langle g_n^2 \rangle = 1$, angular brackets denoting the average over the (common) period $2\pi q$. To study the effect of the two weak harmonic perturbations, one calculates the Lyapunov exponent (LE) for $\varepsilon \neq 0$: $\lambda = \mathrm{Re}\langle \ln[\mu + \varepsilon(f_n + \eta g_n)]\rangle$. For small ε, the LE becomes

$$\lambda = \ln \mu - \frac{1}{2}\left(\frac{\varepsilon}{\mu}\right)^2 (1+\eta^2) + O(\varepsilon^3). \tag{2}$$

To clarify the effect of the second resonant perturbation g_n on the reduction of instabilities (positive LE), let us consider that, in the absence of the second perturbation $(\eta = 0)$, we are in a weakly unstable initial state with $\ln \mu \gtrsim \frac{1}{2}(\varepsilon/\mu)^2$ such that $\lambda \approx \lambda^+(\eta=0) \equiv \ln \mu - \frac{1}{2}(\varepsilon/\mu)^2 \gtrsim 0$. Then, by increasing η, the LE $\lambda = \lambda^+(\eta=0) - \frac{1}{2}(\varepsilon/\mu)^2\eta^2$ decreases and in some case may become negative, thus stabilizing x.

To provide a rigorous formulation of the technique, we shall concentrate here on a simple model for a universal escape situation:

$$\ddot{x} = x - \beta[1 + \eta \sin(\Omega t + \phi)]x^2 - \delta \dot{x} + \gamma \sin(\omega t), \tag{3}$$

where Ω, η, and ϕ are the normalized frequency, amplitude, and initial phase, respectively, of the PE $(\eta \ll 1)$, which will have an inhibitory effect on the chaotic escape of the remaining system $(\delta, \gamma \ll 1)$ [10], and ω, δ, and γ are the normalized parameters of frequency, damping coefficient, and driving term amplitude, respectively.

The rest of the paper is organized as follows. In Sec. II we derive analytical results based on a Melnikov analysis (MA) concerning the ultrasubharmonic resonance case: $\Omega = p\omega/q$, $q > 1$ $(p \neq q)$, p,q positive integers. In Sec. III we apply the results of Sec. II to the case of incommensurate escape-suppressing excitations by considering the asymptotic behavior of a series of associated systems whose escape-suppressing frequencies are ever better rational approximations of the irrational escape-suppressing frequency. In Sec. IV we present numerical evidence supporting the theoretical predictions from previous sections. Finally, Sec. V gives a brief summary of the results.

II. MELNIKOV ANALYSIS

Melnikov analysis [11] has become standard for detecting the splitting of invariant manifolds for a wide variety of dynamical systems close to ''integrable'' systems with associated separatrices. As is well known, its predictions for the appearance of chaos are both approximate (the MA is a perturbative technique) and limited (only valid for orbits starting at points sufficiently near the separatrix). Since MA has been described many times by different authors, we shall not discuss it in detail here, but refer the reader to that literature [11–13]. It is worth mentioning that the criterion for a homoclinic tangency—accurately predicted by MA—in diverse systems [1,14] is coincident with the change from a smooth to an irregular, fractal-like basin boundary [15]. These results connect MA predictions with those concerning the erosion of the basin boundary.

By applying MA to Eq. (3) one straightforwardly obtains the Melnikov function (MF) [2]

$$M(t_0) = -C - A\cos(\omega t_0) + B\cos(\Omega t_0 + \phi), \tag{4}$$

with

$$C = \frac{6\delta}{5\beta^2}, \tag{5}$$

$$A = \frac{6\pi\gamma}{\beta}\omega^2 \operatorname{csch}(\pi\omega),$$

$$B = \frac{3\pi\eta}{5\beta^2}\Omega^2(\Omega^2+1)(\Omega^2+4)\operatorname{csch}(\pi\Omega).$$

As is well known [13], the MF $M(t_0)$ measures the distance between the perturbed stable and unstable manifolds in the Poincaré section at t_0. If $M(t_0)$ has a simple zero, then a homoclinic bifurcation occurs, signifying the possibility of chaotic behavior, i.e., only necessary conditions for steady chaos are obtained from MA, and therefore one always has, *a priori*, the possibility of finding sufficient conditions for the elimination of even transient chaos.

Let us suppose that, in the absence of any escape-suppressing excitation $(B=0)$, the associated MF $M_0(t_0) = -C - A\cos(\omega t_0)$ changes sign at some t_0, i.e., $C \leqslant A$. If we now let the escape-suppressing excitation act on the system such that $B \leqslant A - C$, this relationship represents a sufficient condition for $M(t_0)$ to change sign at some t_0. Thus, a necessary condition for $M(t_0)$ to always have the same sign is $B > A - C \equiv B_{\min}$. It is obvious that for this relationship to be also a sufficient condition for $M(t_0)$ to be negative for all t_0, one must have

$$A - B \geqslant B\cos(\Omega t_0 + \Psi) - A\cos(\omega t_0). \tag{6}$$

In Ref. [2] it is demonstrated that Eq. (6) can only be true for (certain) values of ω, Ω, and ϕ if a resonance condition is satisfied, $p\omega = q\Omega$, for some positive integers p and q. In such a situation, the relationship

$$\frac{p}{q} = \frac{2m+1-\phi/\pi}{2n+1} \tag{7}$$

with m,n non-negative integers is a sufficient condition for Eq. (6) to be satisfied for an infinity of t_0 values. Finally, for the subharmonic case $(q=1)$, $B = B_{\max} \equiv A/p^2$ is a sufficient condition for Eq. (6) to be satisfied for all t_0 [2].

For the ultrasubharmonic case $(q>1,\ p \neq q)$ there exist different conditions [that given by Eq. (7) among them] which are sufficient for Eq. (6) to be satisfied for an infinity of t_0 values but not sufficient for Eq. (6) to be satisfied for all t_0 [i.e., $M(t_0)$ may still present simple zeros]. We choose among them that condition making $M(t_0)$ as near as possible to the tangency condition for $B = B_{\min}$, in the sense that (at least) one of the local maxima of $M(t_0)$ is the lowest:

$$\frac{p}{q} = \frac{2m+1-\phi/\pi}{2n},$$

$$\left(\frac{p}{q}\right)^2 > 1 - \frac{C}{A}, \tag{8}$$

with m,n non-negative integers. This means that, although now chaotic transients cannot be completely eliminated (i.e., homoclinic bifurcations cannot be suppressed), one would expect to have a fair chance of reducing chaotic escape. Figure 1 depicts, as an example, the normalized MF $M'(\tau_0,\phi) \equiv M(t_0)/A$ versus $\tau_0 \equiv \omega t_0$ and ϕ for $B = B_{\min}$ and the resonances $p/q = \{2/3, 3/5\}$. Note that, in each period of M', its local maxima (with respect to τ_0) are the lowest (with respect to ϕ) at the suitable initial phases given by Eq. (8).

Remarks

First, for a given resonance p/q, one has q suitable values of the initial phase which are uniformly distributed in the interval $[0,2\pi]$, $\Delta\phi_{\text{suitable}} \equiv 2\pi/q$ being the gap between any two adjacent suitable initial phases. Note that π is a suitable initial phase for all the resonances p/q. It is worth mentioning that this remarkable property of $\phi_{\text{suitable}} = \pi$ does not hold for any suitable initial phase of the two-well Duffing oscillator considered in Ref. [8]. For such a Duffing oscillator, an upper threshold for the amplitude is also deduced by imposing the condition that the chaos-suppressing excitation may not enhance the initial chaos (cf. Ref. [8]). This upper threshold is pertinent since all the solutions of the Duffing oscillator are bounded. In particular, one can observe chaos confined within one of the wells or chaos around the two wells depending upon the values of the parameters. However, for our escape oscillator (3) (whose associated potential has a single well), one could expect that any value of the amplitude of the escape-suppressing excitation higher than its corresponding lower threshold would have an enhancing

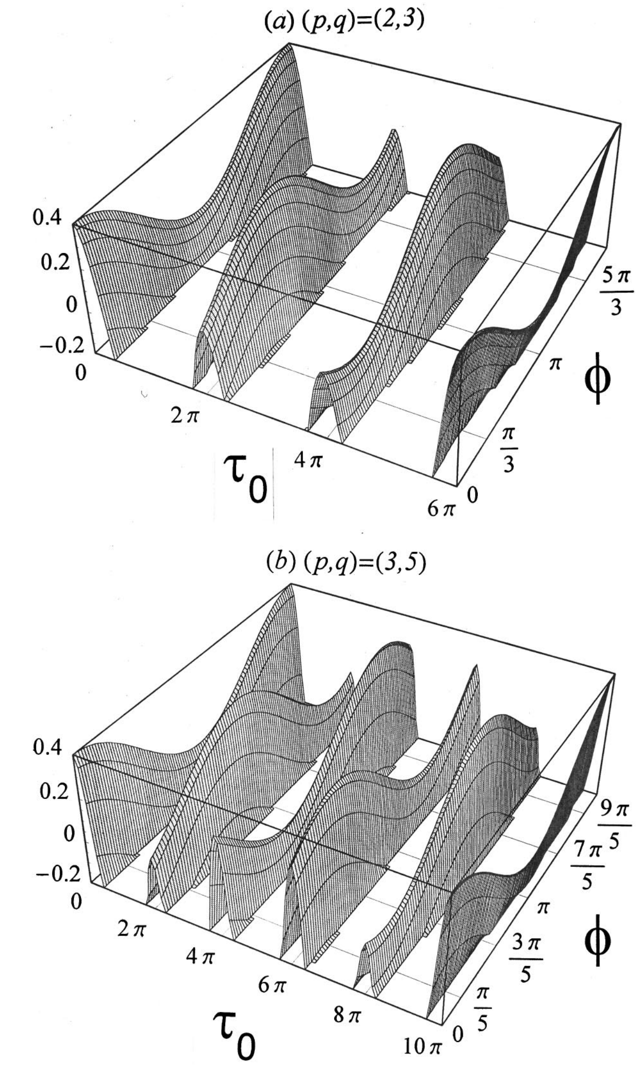

FIG. 1. Normalized Melnikov function $M'(\tau_0,\phi)\equiv M(t_0)/A$ [cf. Eq. (4)] vs τ_0 and ϕ in the range $[-0.2,0.4]$, $\tau_0\equiv\omega t_0$, for $B=B_{\min}=A-C$ and $C/A=0.8$. Resonance $p/q=$(a) 2/3, (b) 3/5. Observe that the local maxima (with regard to τ_0) are the lowest (with regard to ϕ) at π, $5\pi/3$, $\pi/3$ (π, $9\pi/5$, $3\pi/5$, $7\pi/5$, $\pi/5$), respectively, for $p/q=2/3$ ($p/q=3/5$). The variables M', τ_0, and ϕ (rad) are dimensionless.

(or at least not suppressive) effect on the initial escape situation. We shall see in Sec. IV that numerical experiments support this conjecture. Second, $\phi_{\text{suitable}}=0$ ($\phi_{\text{suitable}}=\pi/q$) is the lowest suitable initial phase for all the resonances p/q with q even (odd). Third, the lowest amplitude threshold $\eta_{\min}$ associated with $B_{\min}$ is

$$\eta_{\min}=\left(1-\frac{C}{A}\right)R, \tag{9}$$

with

$$R=\frac{10\beta\gamma\omega^2}{\Omega^2(\Omega^2+1)(\Omega^2+4)}\,\frac{\sinh(\pi\Omega)}{\sinh(\pi\omega)} \tag{10}$$

[cf. Eq. (5)].

III. INCOMMENSURATE ESCAPE-SUPPRESSING EXCITATIONS

We now demonstrate the possibility of reducing chaotic escape by incommensurate PEs (i.e., Ω/ω irrational) from the results of the preceding section. This involves replacing the irrational ratio Ω/ω with approximations derived from continued fractions. This technique has been much used in studying phase-locking phenomena in both dissipative and Hamiltonian systems as well as in characterizing strange nonchaotic attractors in quasiperiodically forced systems [16]. Our approach clearly differs from that based on the application of high-frequency PEs, which is discussed in the framework of an effective averaged nonlinear equation in Ref. [17]. To illustrate the procedure we intentionally choose the golden section $\Omega/\omega=\Phi\equiv(\sqrt{5}-1)/2$, since it is the irrational number that is the worst approximated by rational numbers in the form of continued fractions. As is well known, Φ can be approximated by the sequence of rational numbers $(\Omega/\omega)_k=F_{k-1}/F_k$ where $F_k=1,1,2,3,5,\ldots$, are the Fibonacci numbers such that $\lim_{k\to\infty}(\Omega/\omega)_k=(\sqrt{5}-1)/2$. For each $(\Omega/\omega)_k$ we replace the quasiperiodically excited system (3) [with $\Omega\equiv(\sqrt{5}-1)\omega/2$] by the periodically excited system

$$\ddot{x}=x-\beta\left[1+\eta\sin\left(\frac{F_{k-1}}{F_k}\omega t+\phi\right)\right]x^2-\delta\dot{x}+\gamma\sin(\omega t), \tag{11}$$

giving a sequence of periodically excited systems whose involved frequencies satisfy an ultrasubharmonic resonance condition. Now we can apply the theoretical predictions of Sec. II to each system (11) for increasing values of k. Thus, the corresponding values of the suitable initial phase and amplitude are [cf. Eqs. (8) and (9), respectively]

$$\phi_{\text{suitable},k}=\pi\left(2m+1-2n\frac{F_{k-1}}{F_k}\right)\quad(\text{mod}\,2\pi), \tag{12}$$

$$\eta_{\min,k}=\left(1-\frac{C}{A}\right)R_k, \tag{13}$$

$$R_k=\frac{10\beta\gamma}{(F_{k-1}/F_k)^2[\omega^2(F_{k-1}/F_k)^2+1][\omega^2(F_{k-1}/F_k)^2+4]}\times\frac{\sinh(\pi\omega F_{k-1}/F_k)}{\sinh(\pi\omega)}, \tag{14}$$

where m,n are non-negative integers.

Remarks

First, for fixed ω, the ratio $\eta_{\min,k}/\eta_{\min,\infty}$ converges very quickly to 1 as $k\to\infty$. This means that the values of $\eta_{\min,k}$ associated with early convergents (3/5, 5/8, 8/13,...) are really very close to the limiting value $\eta_{\min,\infty}$ corresponding to Φ. Second, the successive convergents of Φ (1/1, 1/2, 2/3, 3/5, 5/8, 8/13, 13/21, 21/34, 34/55,...) present one even denominator q for every two odd, and thus whether 0 is or is not one of the associated suitable initial phases depends upon the parity of q (cf. the second remark of Sec. II). Also π is a suitable initial phase for *all* the convergents (cf. the first remark of Sec. II). Third, in contrast to the aforementioned asymptotic behavior of the amplitudes $\eta_{\min,k}$, the number of suitable initial phases $\phi_{\mathrm{suitable},k}$ tends to infinity as $k\rightarrow\infty$, and for each rational approximation to Φ, given by a certain convergent, the corresponding values of $\phi_{\mathrm{suitable},k}$ are uniformly distributed in the interval $[0, 2\pi]$. Since two successive convergents F_{k-1}/F_k, F_k/F_{k+1} differ only in ever higher decimal places as $k\rightarrow\infty$, one should expect that the suppressive effective (observed) values of the initial phase $\phi_{\mathrm{effective}}$, corresponding to a certain convergent, would be related not only to its corresponding values $\phi_{\mathrm{suitable},k}$ but also to the suitable initial phases associated with its preceding convergents. In particular, the effective values $\phi_{\mathrm{effective}}$ should correspond to the points where the suitable initial phases associated with the chosen convergent and its precedents concentrate. One expects this prediction to gain in accuracy as $k\rightarrow\infty$.

IV. REDUCTION OF THE EROSION OF NONESCAPING BASINS

For the escape model (3), the initial conditions will determine, for a fixed set of its parameters, whether the system escapes to an attractor at infinity (with $x\rightarrow\infty$ as $t\rightarrow\infty$), or settles into a bounded oscillation. As is well known [1], there can exist a rapid and dramatic erosion of the safe basin (union of the basins of the bounded attractors) due to encroachment by the basin of the attractor at infinity (escaping basin). We shall show in the following how the erosion of the safe basin is reduced under the theoretical conditions established above. To generate the basins of attraction numerically, we select a grid of (uniformly distributed) 300×300 starting points in the region of phase space $\{x(t=0)\in[0,1.8],\ \dot{x}(t=0)\in[-0.8,0.8]\}$. From this grid of initial conditions, each integration is continued until either x exceeds 20, at which point the system is deemed to have escaped (i.e., to the attractor at infinity), or the maximum allowable number of cycles, here 20, is reached. In the absence of an escape-suppressing excitation $(\eta=0)$, we assume that the system presents a dramatic erosion and stratification of the basin (as in the example shown in Fig. 2 where the color white represents the nonescaping basin and black the escaping basin). For the set of parameters considered in Fig. 2 $(\beta=1,\delta=0.1,\gamma=0.08,\omega=0.85)$, we calculated, for each resonance p/q, the escape probability normalized to that of the corresponding case with no escape-suppressing excitation, $P(\eta)/P(\eta=0)$, versus the initial phase ϕ for several values of η. Typically we found that no reduction of initial escape is attained for $\eta>\eta_{\min}$ and arbitrary initial phase, with $\eta_{\min}$ given by Eq. (9). For small amplitudes $(\eta\lesssim\eta_{\min})$, the probability $P(\eta\neq0)$ becomes lower than $P(\eta=0)$ over short ranges of ϕ which are typically centered on (some of) the predicted suitable initial phases (in the aforementioned sense) for each resonance p/q. As an example, Fig. 3 shows $P(\eta)/P(\eta\neq0)$ vs ϕ for the resonance $p/q=3/4$ and the amplitudes $\eta=\{0.025,0.015,0.010,0.008\}$. The theoretical predictions are $\eta_{\min}\approx0.0242$ and $\phi_{\mathrm{suitable}}=\{0,\pi/2,\pi,3\pi/2\}$ [cf. Eqs. (5) and (8)]. One sees that the normalized probability presents minima, as a function of the initial phase, that come progressively closer to the predicted suitable values as the amplitude decreases. The reduction of chaotic escape is achieved for $\eta=0.008$ over a range centered on the initial phase $\phi=\phi_{\mathrm{suitable}}\equiv3\pi/2$. A further example is shown in Fig. 4 for the resonance $p/q=8/9$ and $\eta=\{0.08,0.03,0.022\}$. In this case, the theoretical predictions are $\eta_{\min}\approx0.0217$ and $\phi_{\mathrm{suitable}}=\{\pi/9,\pi/3,5\pi/9,7\pi/9,\pi,11\pi/9,13\pi/9,15\pi/9,17\pi/9\}$. For $\eta=0.08$ the normalized probability reaches its maximum value (~1.34) for any initial phase, which represents the situation where the nonescaping basin has been completely destroyed (for the resolution considered here). For $\eta<0.08$ the normalized probability solely presents minima at the val-

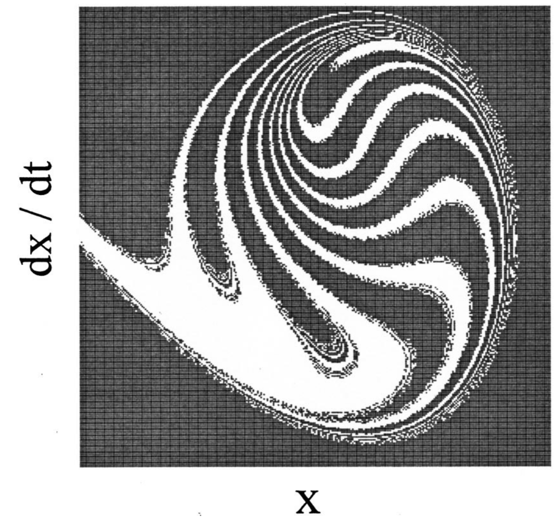

FIG. 2. Erosion basin of the system (3) for $\beta=1$, $\delta=0.1$, $\omega=0.85$, $\gamma=0.08$, and $\eta=0$ in the window $0<x<1.8$, $-0.8<\dot{x}<0.8$.

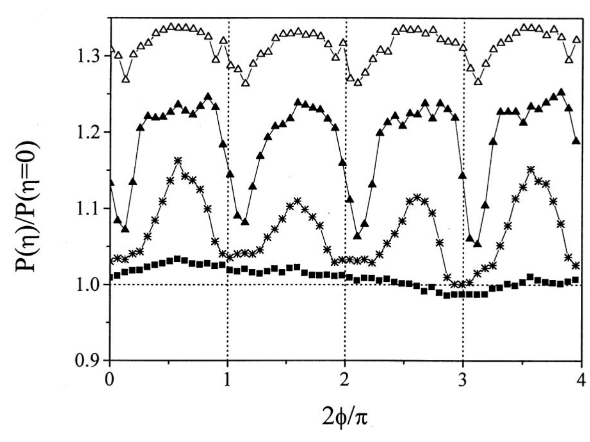

FIG. 3. Normalized escape probability (dimensionless quantity, see the text) vs ϕ (rad) for four values of the amplitude: $\eta=0.025$ (△), 0.015 (▲), 0.01 (★), and 0.008 (■). System parameters are $\beta=1$, $\delta=0.1$, $\gamma=0.08$, $\omega=0.85$, and $p/q=3/4$. Solid lines are plotted solely to guide the eye. Dotted vertical lines indicate the predicted suitable initial phases.

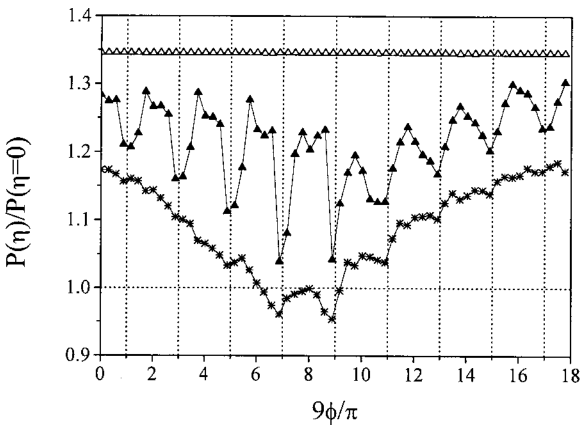

FIG. 4. Normalized escape probability (dimensionless quantity, see the text) vs ϕ (rad) for three values of the amplitude: $\eta=0.08$ (△), 0.03 (▲), and 0.022 (★). $p/q=8/9$, and the remaining parameters are as in Fig. 3. Solid lines are plotted solely to guide the eye. Dotted vertical lines indicate the predicted suitable initial phases.

ues predicted above. One sees that for $\eta=0.022$ ($\approx\eta_{\mathrm{min}}$) the reduction of chaotic escape is achieved over two ranges of the initial phase centered on each of the suitable initial phases $\{7\pi/9,\pi\}$.

Consider now the case of incommensurate escape-suppressing excitations. The only "off-resonance" excitations that can be numerically considered are those with irrational frequencies to the limits of computational precision. As in the previous examples of arbitrary resonances p/q, we found—for sufficiently small amplitudes ($\eta\lesssim\eta_{\mathrm{min}}$)—that the overall behavior of the normalized escape probability presents a minimum near $\phi=\phi_{\mathrm{suitable}}=\pi$ (see Fig. 4), as expected (cf. the first remark in Sec. II). Figure 5 shows $P(\eta)/P(\eta=0)$ vs ϕ for the amplitude $\eta=0.010$ and three values of p/q: the two convergents 3/5, 8/13, and 0.618 033 988 749, this last being Φ to the limit of computational precision considered here. The theoretical predictions for the threshold amplitude are $\eta_{\mathrm{min}}\approx0.028\,57$, 0.028 00, and 0.027 91, respectively [cf. Eqs. (13) and (14)], which are very close as predicted (cf. the first remark in Sec. III). In Fig. 5(a), there are some clear-cut additional minima—which are not associated with any of the corresponding suitable initial phases $\phi_{\mathrm{suitable}}=\{\pi/5,3\pi/5,\pi,7\pi/5,9\pi/5\}$—that are absent from the two lower plots in Fig. 4. Typically, the number of such additional "off-prediction" minima increases with increasing convergent order, as can be seen for $p/q=8/13$ in Fig. 5(b). This confirms (see also Fig. 6 for additional examples) the prediction in the third remark in Sec. III. It is also observed that, in general, the plot of the normalized escape probability is asymmetric with respect to the particular suitable initial phase $\phi_{\mathrm{suitable}}=\pi$: the ranges where reduction of chaotic escape occurs are larger for $\phi<\pi$ than for $\phi>\pi$ (see Figs. 5 and 6). Figure 6 gives plots of the normalized escape probability vs the initial phase for the convergent $p/q=34/55$ and the amplitudes $\eta=\{0.010,0.015,0.020\}$. We typically found that the overall behavior of $P(\eta)/P(\eta=0)$ vs ϕ for a fixed amplitude η

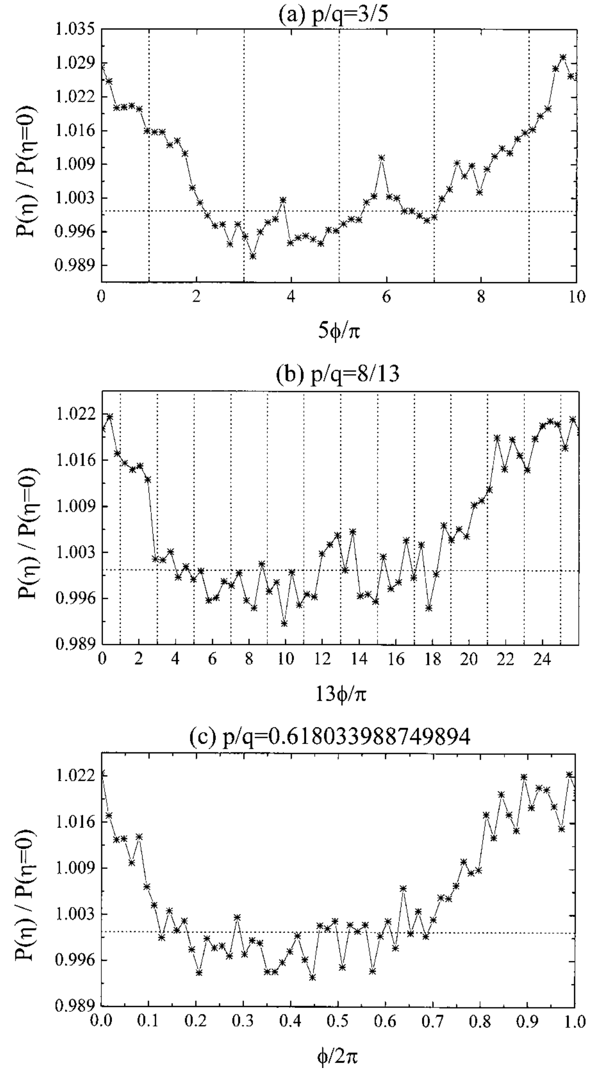

FIG. 5. Normalized escape probability (dimensionless quantity, see the text) vs ϕ (rad) for $\eta=0.01$, $\beta=1$, $\delta=0.1$, $\gamma=0.08$, and $\omega=0.85$. $p/q=$(a) 3/5, (b) 8/13, and (c) 0.618 033 988 749 894 [$\simeq(\sqrt{5}-1)/2$]. Solid lines are plotted solely to guide the eye. Dotted vertical lines indicate the predicted suitable initial phases.

(and also the remaining parameters fixed) is quite similar for all the successive convergents beginning with the third convergent or so, as can be appreciated by comparison of Figs. 5(b), 5(c), and 6(c). This is coherent with the insensitivity of η_{min} with regard to the convergent order (cf. the first remark in Sec. III). Figure 6(b) [and to a lesser degree Fig. 6(c)] shows that the overall behavior of $P(\eta)/P(\eta=0)$ presents a few dominant minima at certain effective initial phases $\phi_{\mathrm{effective}}$ (cf. the third remark in Sec. III), which cannot be explained solely on the basis of the predicted suitable initial phases $\phi_{\mathrm{suitable}}=\{(2n-1)\pi/55,\ n=1,...,55\}$. In order to test the conjecture about the origin of the effective initial phases (cf. the third remark in Sec. III), we calculated the histogram shown in Fig. 7, which is a plot of the density of suitable initial phases associated with the complete series of convergents of Φ up to the ninth order $p/q=\{1/1,1/2,2/3,3/5,5/8,8/13,13/21,21/34,34/55\}$ vs the initial phase. One observes three remarkable properties. First, the

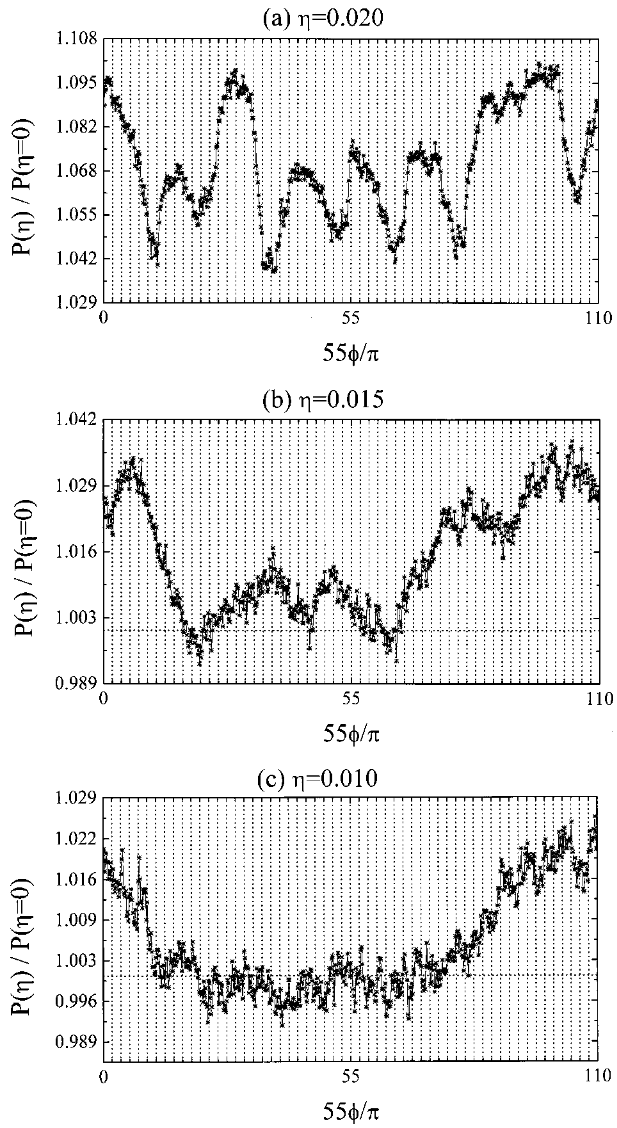

FIG. 6. Normalized escape probability (dimensionless quantity, see the text) vs ϕ (rad) for the convergent $p/q=34/55$, and the remaining parameters as in Fig. 3. $\eta=$(a) 0.020, (b) 0.015, and (c) 0.010. Solid lines are plotted solely to guide the eye. Dotted vertical lines indicate the predicted suitable initial phases.

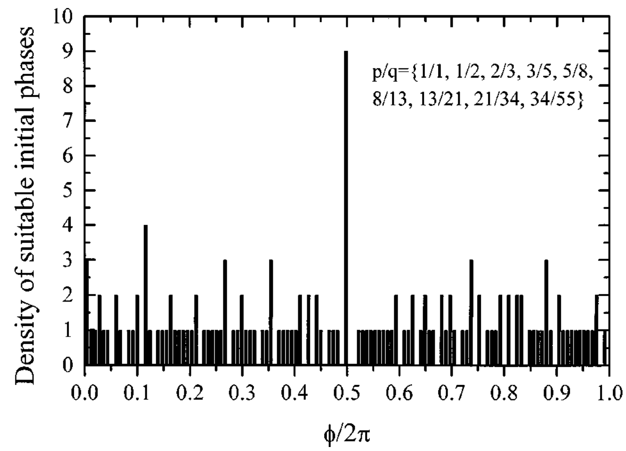

FIG. 7. Histogram showing the density of suitable initial phases (dimensionless quantity, see the text) for the first nine convergents of the golden ratio, vs ϕ (rad).

histogram is asymmetric with respect to the central value $\phi=\pi$: the number of maxima of the density function is larger for $\phi<\pi$ than for $\phi>\pi$. This is coherent with the aforementioned asymmetry of the distribution of ranges where a reduction of chaotic escape occurs. Second, the density function has its main maxima (density $\geqslant 3$) at and only at initial phases that roughly coincide with the aforementioned effective initial phases $\phi_{\text{effective}}$ [compare Figs. 6(c) and 7]. Third, the density function has its highest maximum at the particular suitable initial phase $\phi_{\text{suitable}}=\pi$ as predicted (cf. the second remark in Sec. III). Such a maximum is the most isolated in the sense that the range of initial phases around π with null density is the largest. This is coherent with the fact that the reduction of chaotic escape expected at $\phi_{\text{suitable}}=\pi$ is hardly observable, it being sharply localized when in fact it does occur [see, e.g., Figs. 5(b) and 5(c)]. It is expected that the mechanism discussed for the reduction of chaotic escape by incommensurate PEs will work for amplitudes sufficiently lower than the theoretically predicted threshold amplitudes.

V. CONCLUSION

In sum, we have shown that the effectiveness of a PE that satisfies an ultrasubharmonic resonance condition with the escape-inducing excitation in reducing chaotic escape from a potential well strongly depends on its initial phase. Analytical estimates of the suitable initial phases and amplitudes for reducing the chaotic escape were found by means of MA. In addition, the reduction of chaotic escape by applying small-amplitude incommensurate PEs was demonstrated from the findings for ultrasubharmonic resonances. For this case, the proposed reduction mechanism demonstrates the great complexity of the role played by the initial phase of the PE. In general, numerical results based on a high-resolution grid of initial conditions showed excellent agreement with the theoretical predictions of the escape-reducing initial phases, while the values obtained numerically of the amplitude for escape reduction were (in some cases) lower than those predicted from MA. We should emphasize that the theoretical approach we have discussed as well as the conclusion that chaos can be reduced (or even completely suppressed) by incommensurate PEs are both general enough to be applied to many other dissipative nonautonomous systems.

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